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Diffusion on multifractals

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Abstract

We introduce a class of random fields with variable mean-square regularity order defined on multifractal domains. The local singularity order of these random fields then depends on the initial variable mean-square regularity order, and on the variable local singularity exponent of the multifractal measure defining the local dimension of the domain considered. The theory is developed in a generalized framework through the covariance factorization, using the tools of reproducing kernel Hilbert spaces and fractional Sobolev spaces of variable order.

1 Introduction

Anomalous diffusion in disordered media is a rapidly growing area of scientific research which has applications in a wide range of fields including geophysics, hydrology, biological systems, protein dynamics and mathematical finance (Levin 1994, Barabasi and Stanley 1995, Shlesinger *et al.* 1995, Iannaccone and Khokha 1996, Mandelbrot 1997, Falconer 1997, Carpinteri and Mainardi 1997, Hilfer 2000, Schlichter *et al.* 2000). Some typical examples include heat conduction and fluid flow in porous media, relaxation in synthetic or biopolymers, propagation of seismic waves, diffusion and transport of macromolecules in living tissues, conformational relaxation and diffusion of a protein in an energy landscape, option pricing in a Black-Scholes market with frictions. The two major reports by Bouchaud and Georges (1990) and Metzler and Klafter (2000) provide comprehensive reviews of fractional/anomalous diffusion and an extensive collection of examples from a variety of application areas. Anh and Leonenko (2001) listed a number of articles on fractional diffusion from the statistical point of view. Angulo *et al.* (2000) introduced a class of fractional heat equations whose solutions exhibit spatial and temporal long-range dependence. Anh and Leonenko (1999, 2000, 2001, 2002a, 2002b, 2003) developed a theory for spectral analysis, renormalisation and homogenisation of fractional kinetic equations and fractional diffusion-wave equations, leading to their scaling laws and non-Gaussian central limit theory. Higher-order information contained in these equations has also been extracted and shown to be useful for modelling a wide range of nonlinear and non-Gaussian systems such as bilinear stochastic systems, sky-wave radar signals with Doppler spreading, non-Gaussian signals embedded in Gaussian noise (Anh, Leonenko and Sakhno 2003, 2004]).

Many models for *diffusion in fractal media* have been adapted from Brownian motion-type processes developed in probability theory, essentially extending the traditional Brownian motion on a regular lattice to Brownian motion on geometric fractals such as the Sierspinski gasket, the Sierspinski carpet, post-critically finite self-similar sets (Barlow and Bass 1992, Lindstrom 1993, Kigami 1993, Dobrushin and Kusuoka 1993, Barlow 1998). The generator of the derived Brownian motion is then defined as the "Laplacian" on these fractals. This approach yields diffusion operators, but places severe restrictions on the nature of the underlying fractal sets. A related but more successful approach has been the derivation of Dirichlet forms, which give the Laplacian directly (Jonsson 2000, Kigami 2001). Kumagai (2002), Grigoryan *et al.* (2003) and Zähle (2003) investigated the Laplacian, Riesz potential and heat kernel on general fractal sets from the angle of function spaces on fractals.

The development of stochastic models in connection with the *multifractal formalism* is also extensive. *Multifractal models* were originally introduced to study strange attractors (Hentschel and Procaccia 1983, Halsey *et al.* 1986) and the spatial distribution of the kinetic energy dissipation rate in fully developed turbulence (Mandelbrot 1974, Frisch and Parisi 1985). They have subsequently been used in many areas of applied sciences such as mathematical finance (Mandelbrot *et al.* 1997), genomics (Anh *et al.* 2001, 2002, Yu *et al.* 2001, 2003). Harte (2001) and Riedi (2003) contain an extensive bibliography of the subject. A variety of multiplicative cascades and iterated function systems has been shown to generate multifractals (Mandelbrot 1974, Kahane and Peyrière 1976, Barnsley and Demko 1985, Frisch 1995, Falconer 1997, Lau *et al.* 2001). Brownian motion in multifractal time and most Lévy processes are also known to have multifractal paths (Jaffard 1999, Riedi 2003).

The multifractal formalism, essentially based on the singularity spectrum (Jaffard 1999), provides a useful tool in practice to estimate multifractal properties. However, it does not lead to a unique characterisation of the multifractal process underlying a set of data. A radical approach in this direction is developed in Ruiz-Medina, Anh and Angulo (2001, 2004a) and Ruiz-Medina, Angulo and Anh (2002, 2003a), which leads to a class of models for *diffusion on multifractals* generated by pseudodifferential operators of variable order (Ruiz-Medina, Anh and Angulo 2004b). In this paper we outline the development of this theory. This work introduces a class of random fields with heterogeneous quadratic variation defined on multifractal domains. A key component here is the development of analytical results for the trace operator on a multifractal domain of a fractional Sobolev space of variable order via its atomic decomposition. Such results allow the connection between the multifractal geometry of the domain and the regularity property of functions in fractional Sobolev spaces. The function defining the singular exponent of the variogram is affected by the multifractal geometry of the domain, ensuring the non-triviality of the singularity spectrum, hence the multifractality of the random field.

2 Random fields of variable regularity order

This section introduces a class of multifractional models generated by pseudodifferential operators of variable order. Some prototypes of this class are multifractional Brownian motion, multifractional free Markov fields (extending free Markov fields, which play a basic role in quantum field theory (Dobrushin 1979, p. 280)) and multifractional Riesz-Bessel motion.

Benassi *et al.* (1997) considered Gaussian random fields whose covariance function is defined in terms of an elliptic pseudodifferential operator. Markov processes associated with pseudodifferential operators with smooth symbols were studied by Bass (1988), Jacob and Leopold (1993), Jacob (1994), Komatsu (1995), Kikuchi and Negoro (1997), for example. In particular, Bass (1988) considered the generator $-\left(-\Delta\right)^{\sigma(x)/2}$, where Δ is the Laplacian, for a continuous function $\sigma\left(x\right)$ satisfying $0 < \sigma\left(x\right) < 2$ and called the generated process an isotropic stable-like process. Komatsu (1995) extended this class to generalized stable-like processes. Jacob and Leopold (1993) showed that there exists a Feller semigroup generated by the pseudodifferential operator whose symbol is the function $-\left(1+|\xi|^2\right)^{\sigma(x)}$, $0 < \inf\, \sigma \leq \sup\, \sigma \leq 2$. Kikuchi and Negoro (1997) extended the result to strongly elliptic pseudodifferential operators with suitable variable order.

We introduce in this section a class of random fields with variable local singularity order using the theory of generalized random fields on fractional Sobolev spaces of variable order. A pseudoduality condition is formulated to obtain their covariance factorization, as well as their white-noise linear filter representation in terms of pseudodifferential operators of variable order.

Let δ and ρ be real numbers with $0 \leq \delta < \rho \leq 1$, σ be a real-valued function in $\mathcal{B}^{\infty}(\mathbb{R}^n)$, the set of all C^{∞}- functions whose derivatives of all orders are bounded. We say that a function $p(\mathbf{x},\boldsymbol{\xi}) \in \mathcal{B}^{\infty}\left(\mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}^n_{\boldsymbol{\xi}}\right)$ belongs to $\mathcal{S}^{\sigma}_{\rho,\delta}$ if and only if for any multi-indices α and β there exists some positive constant $C_{\alpha,\beta}$ such that

$$|D^{\alpha}_{\boldsymbol{\xi}} D^{\beta}_{\mathbf{x}} p(\mathbf{x},\boldsymbol{\xi})| \leq C_{\alpha,\beta} \langle \boldsymbol{\xi} \rangle^{\sigma(\mathbf{x})-\rho|\alpha|+\delta|\beta|}, \tag{2.1}$$

where $D^{\alpha}_{\boldsymbol{\xi}}$ and $D^{\beta}_{\mathbf{x}}$ respectively denote the derivatives with respect to $\boldsymbol{\xi}$ and $\mathbf{x}$, and $\langle \boldsymbol{\xi} \rangle = (1+|\boldsymbol{\xi}|^2)^{1/2}$. The following semi-norm is considered for the elements of $\mathcal{S}^{\sigma}_{\rho,\delta}$:

$$|p|^{(\sigma)}_{l} = \max_{|\alpha+\beta|\leq l}\, \sup_{(\mathbf{x},\boldsymbol{\xi})\in\mathbb{R}^n\times\mathbb{R}^n} \left\{ |D^{\alpha}_{\boldsymbol{\xi}} D^{\beta}_{\mathbf{x}} p(\mathbf{x},\boldsymbol{\xi})| \langle \boldsymbol{\xi} \rangle^{-\sigma(\mathbf{x})+\rho|\alpha|-\delta|\beta|} \right\}.$$

Definition 1 *(Kikuchi and Negoro 1995, 1997) For $u \in \mathcal{S}(\mathbb{R}^n)$, the set of rapidly decreasing functions of the Schwartz space, and $p \in \mathcal{S}^{\sigma}_{\rho,\delta}$, let $P : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ be defined as*

$$Pu(\mathbf{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\mathbf{x}\boldsymbol{\xi}} p(\mathbf{x},\boldsymbol{\xi}) \hat{u}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{2.2}$$

where $\hat{u}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} e^{-i\mathbf{x}\boldsymbol{\xi}} u(\mathbf{x})d\mathbf{x}$ *is the Fourier transform of* u. *We refer to* $P = p(\mathbf{x}, D_{\mathbf{x}})$ *as a pseudodifferential operator of variable order with symbol* $p \in \mathcal{S}^{\sigma}_{\rho,\delta}$. *The set of all pseudodifferential operators with symbol* p *of the class* $\mathcal{S}^{\sigma}_{\rho,\delta}$ *is denoted by* $\boldsymbol{\mathcal{S}}^{\sigma}_{\rho,\delta}$.

A pseudodifferential operator $P \in \boldsymbol{\mathcal{S}}^{\sigma}_{\rho,\delta}$ is elliptic if there exist $c > 0$ and $M > 0$ such that

$$|p(\mathbf{x}, \boldsymbol{\xi})| \geq c\langle \boldsymbol{\xi} \rangle^{\sigma(\mathbf{x})}, \quad \text{for } |\boldsymbol{\xi}| \geq M. \tag{2.3}$$

We now give the definition of fractional Sobolev spaces of variable order.

Definition 2 *Let* σ *be a real-valued function in* $\mathcal{B}^{\infty}(\mathbb{R}^n)$. *The Sobolev space of variable order* σ *on* $\mathbb{R}^n$ *is defined as*

$$H^{\sigma(\cdot)}(\mathbb{R}^n) = \left\{ u \in H^{-\infty} = \bigcup_{s\in\mathbb{R}} H^s(\mathbb{R}^n) : \ \langle D_{\mathbf{x}} \rangle^{\sigma(\mathbf{x})} u \in L^2(\mathbb{R}^n) \right\},$$

where

$$\langle D_{\mathbf{x}} \rangle^{\sigma(\mathbf{x})} u = \int_{\mathbb{R}^n} (2\pi)^{-n} \exp(i\mathbf{x}\boldsymbol{\xi}) \langle \boldsymbol{\xi} \rangle^{\sigma(\mathbf{x})} \hat{u}(\boldsymbol{\xi}) d\boldsymbol{\xi},$$

and

$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : \ \langle D_{\mathbf{x}} \rangle^{s} u \in L^2(\mathbb{R}^n) \right\}.$$

The following result characterizes the class of generalized random fields of variable order.

Theorem 1 *(Kikuchi and Negoro 1997) Let* $P \in \boldsymbol{\mathcal{S}}^{\sigma}_{\rho,\delta}$ *be elliptic. Then,*

$$H^{\sigma(\cdot)}(\mathbb{R}^n) = \left\{ u \in H^{-\infty}(\mathbb{R}^n) : \ Pu \in L^2(\mathbb{R}^n) \right\} \tag{2.4}$$

as a set. Moreover, the norm $\|u\|_{H^{\sigma(\cdot)}}(\mathbb{R}^n)$ *is equivalent to the norm*

$$\|u\|_{H^{\sigma(\cdot),P}}(\mathbb{R}^n) := \left(\|Pu\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{H^{\underline{\sigma}}(\mathbb{R}^n)} \right)^{1/2}. \tag{2.5}$$

We next formulate the concept of fractional generalized random field of variable order (FGRFVO), and a pseudoduality condition relative to this concept. The existence of a dual random field implies the ellipticity of the covariance operator of a FGRFVO, that is, its bicontinuity holds with respect to the norm of a Sobolev space of variable order. The introduction of generalized random fields with variable mean-square fractional singularity order allows the definition of ordinary and improper random fields governed by multifractional differential, integral or integro-differential equations. In particular, multifractal Gaussian random fields can be introduced in this framework, extending multifractional Brownian motion.

Let $(\Omega, \ A, P)$ be a complete probability space and let $L^2(\Omega, \ A, P)$ be the Hilbert space of real-valued zero-mean random variables defined on $(\Omega, \ A, P)$ with finite second-order moments and with the inner product defined by

$$< X, Y >_{\mathcal{L}^2(\Omega)} = E[XY], \quad X, Y \in \ \mathcal{L}^2(\Omega \ \mathcal{A}, P). \tag{2.6}$$

Definition 3 *Let* $\alpha(\cdot)$ *be a real-valued function in* $B^{\infty}(\mathbb{R}^n)$, *and let* $X_{\alpha(\cdot)}$ *be defined from* $H^{\alpha(\cdot)}(\mathbb{R}^n)$ *into* $L^2(\Omega, \ A, P)$. *We say that* $X_{\alpha(\cdot)}$ *is a fractional generalized random field of variable order* $\alpha(\cdot)$ *if it is linear and continuous, in the mean-square sense, with respect to the norm defined on* $H^{\alpha(\cdot)}(\mathbb{R}^n)$.

We consider the Hilbert space $H(X_{\alpha(\cdot)})$ defined as the closed span in the $L^2(\Omega, A, P)-$topology of the random components of $X_{\alpha(\cdot)}$, with the norm generated by the inner product (2.6). The covariance function $B_{\alpha(\cdot)}$ of an FGRFVO defines a positive, symmetric and continuous pseudodifferential operator of variable order $R_{\alpha(\cdot)} : H^{\alpha(\cdot)}(\mathbb{R}^n) \longrightarrow H^{-\alpha(\cdot)}(\mathbb{R}^n)$ by the identity

$$B_{\alpha(\cdot)}(\phi, \psi) = E\left[X_{\alpha(\cdot)}(\phi) X_{\alpha(\cdot)}(\psi) \right] = R_{\alpha(\cdot)}(\phi)(\psi) = \langle \left[R_{\alpha(\cdot)}(\psi) \right]^*, \phi \rangle_{H^{\alpha(\cdot)}(\mathbb{R}^n)}. \tag{2.7}$$

We refer to $R_{\alpha(\cdot)}$ as the covariance operator of $X_{\alpha(\cdot)}$, which generates the reproducing kernel Hilbert space (RKHS) $\mathcal{H}(X_{\alpha(\cdot)})$ of $X_{\alpha(\cdot)}$ composed of the functions of $[H^{\alpha(\cdot)}(\mathbb{R}^n)]^*$ defined from the elements of $H(X_{\alpha(\cdot)})$ as

$$f(\phi) = E\left[XX_{\alpha(\cdot)}(\phi)\right], \quad \forall \phi \in H^{\alpha(\cdot)}(\mathbb{R}^n),\ X \in H(X_{\alpha(\cdot)}). \tag{2.8}$$

The RKHS $\mathcal{H}(X_{\alpha(\cdot)})$ is isometric to the dual of the Hilbert space $H(X_{\alpha(\cdot)})$; thus, each function in this space defines an element of the dual of $H(X_{\alpha(\cdot)})$. For fractional generalized random fields of fixed order, the duality condition considered in Ruiz-Medina *et al.* (2001, 2002, 2003a) allows to identify the elements of the RKHS with the random components of a dual generalized random field, which then defines the elements of the dual of the associated Hilbert space of random variables. Such condition is equivalent to the existence of the inverse of the covariance operator. In this paper, we consider a *pseudoduality condition* which is equivalent to the existence of a parametrix for the isometric isomorphism relating $H(X_{\alpha(\cdot)})$ to the RKHS $\mathcal{H}(X_{\alpha(\cdot)})$ (see Equation (2.12) below).

Definition 4 *Let $\alpha(\cdot)$ be as defined in Definition 3. We say that the generalized random field $\widetilde{X}_{\alpha(\cdot)}$: $\left[H^{\alpha(\cdot)}(\mathbb{R}^n)\right]^* \to \mathcal{L}^2(\Omega, \mathcal{A}, P)$ is a* pseudodual generalized random field of variable order *(DGRFVO) for the FGRFVO $X_{\alpha(\cdot)}$ if the following conditions hold:*
(i) $\widetilde{X}_{\alpha(\cdot)}$ is linear and continuous in the mean-square sense with respect to $\left[H^{\alpha(\cdot)}(\mathbb{R}^n)\right]^*$.
(ii) The space $H\left(\widetilde{X}_{\alpha(\cdot)}\right)$ coincides with the space $H\left(X_{\alpha}(\cdot)\right)$.
(iii) For all $\phi \in H^{\alpha(\cdot)}(\mathbb{R}^n)$ and $g \in \left[H^{\alpha(\cdot)}(\mathbb{R}^n)\right]^*$, the inner product
$\left\langle X_{\alpha(\cdot)}(\phi), \widetilde{X}_{\alpha(\cdot)}(g)\right\rangle_{H\left(X_{\alpha(\cdot)}\right)}$ is given by

$$\begin{aligned}\left\langle X_{\alpha(\cdot)}(\phi), \widetilde{X}_{\alpha(\cdot)}(g)\right\rangle_{H\left(X_{\alpha(\cdot)}\right)} &= \widetilde{X}_{\alpha(\cdot)}(g)\left[X_{\alpha(\cdot)}(\phi)\right] = \left[(I+R)g\right](\phi) \\ &= \left[(I+R)^*\phi\right](g),\end{aligned} \tag{2.9}$$

where $R \in \boldsymbol{\mathcal{S}}^{-\infty}_{\rho,\delta} = \bigcap_{m\in\mathbb{R}} \boldsymbol{\mathcal{S}}^{m}_{\rho,\delta}$, for certain δ and ρ with $0 \leq \delta < \rho \leq 1$. Here, $g^* \in H^{\alpha(\cdot)}(\mathbb{R}^n)$ denotes the dual element of g in the Hilbert space $H^{\alpha(\cdot)}(\mathbb{R}^n)$, and for an operator A, A^* denotes the formal adjoint of A.

In the following development, we will also consider the Hilbert space

$$H(\widetilde{X}_{\alpha(\cdot)}) = \overline{\mathrm{sp}}^{\mathcal{L}^2(\Omega,\mathcal{A},P)}\left\{\widetilde{X}_{\alpha(\cdot)}(g) :\ g \in \left[H^{\alpha(\cdot)}(\mathbb{R}^n)\right]^*\right\}, \tag{2.10}$$

and the associated reproducing kernel Hilbert space $\mathcal{H}(\widetilde{X}_{\alpha(\cdot)})$, isometric to the dual space of $H(\widetilde{X}_{\alpha(\cdot)})$, composed of the functions $\phi \in H^{\alpha(\cdot)}(\mathbb{R}^n)$ satisfying

$$\phi(g) = E\left[Y\widetilde{X}_{\alpha(\cdot)}(g)\right], \text{ for a certain } Y \in H(\widetilde{X}_{\alpha(\cdot)}). \tag{2.11}$$

Let $J_{\alpha(\cdot)}$ and $J'_{\alpha(\cdot)}$ be the isometric isomorphisms

$$J_{\alpha(\cdot)} : H\left(X_{\alpha(\cdot)}\right) \to \mathcal{H}\left(X_{\alpha(\cdot)}\right) \text{ and } J'_{\alpha(\cdot)} : H\left(\widetilde{X}_{\alpha(\cdot)}\right) \to \mathcal{H}\left(\widetilde{X}_{\alpha(\cdot)}\right) \tag{2.12}$$

defined as

$$Y \to J_{\alpha(\cdot)}Y, \quad \text{with } \left(J_{\alpha(\cdot)}Y\right)(\phi) = EYX_{\alpha(\cdot)}(\phi), \quad \forall \phi \in H^{\alpha(\cdot)}(\mathbb{R}^n),$$

$$Z \to J'_{\alpha(\cdot)}Z, \quad \text{with } \left(J'_{\alpha(\cdot)}Z\right)(g) = EZ\widetilde{X}_{\alpha(\cdot)}(g), \quad \forall g \in \left[H^{\alpha(\cdot)}(\mathbb{R}^n)\right]^*.$$

Proposition 2 *Assume that the pseudoduality condition holds. Then $R_{X_{\alpha(\cdot)}} = J_{\alpha(\cdot)}J^*_{\alpha(\cdot)}$ and $R_{\widetilde{X}_{\alpha(\cdot)}} = J'_{\alpha(\cdot)}[J'_{\alpha(\cdot)}]^*$*

A proof of Proposition 2 is provided in Ruiz-Medina, Anh and Angulo (2004a). Define $L_{\alpha(\cdot)} = J'_{\alpha(\cdot)}J_0^{-1}$,

$\widetilde{L}_{\alpha(\cdot)} = J_{\alpha(\cdot)} J_0^{-1}$ with J_0 being the isometric isomorphism between $H(\varepsilon)$ and $\mathcal{H}(\varepsilon) = L^2(\mathbb{R}^n)$, where ε represents generalized white noise on $L^2(\mathbb{R}^n)$, that is, a FGRF on $L^2(\mathbb{R}^n)$ satisfying

$$E\left[\varepsilon(f)\varepsilon(g)\right] = \langle f, g\rangle_{L^2(\mathbb{R}^n)}, \quad \forall f, g \in L^2(\mathbb{R}^n),$$

with its RKHS then given by $L^2(\mathbb{R}^n)$.

Under the conditions of the next Proposition, the mean-square Hölder continuity of the class of corresponding ordinary random fields is established (see Theorem 4). The mean-square local Hölder exponents defining the regularity of the ordinary solution to Equation (2.13) are given in terms of the function $\alpha(\cdot)$. We will write $\underline{-\alpha} = \inf_{\mathbf{x}\in\mathbb{R}^n} (-\alpha(\mathbf{x}))$ in the following.

Proposition 3 *Assume that the conditions of Proposition 2 hold, and that $\underline{-\alpha} > n/2$. Then, there exists a unique mean-square continuous ordinary random field $\mathcal{X}_{\alpha(\cdot)}$ satisfying*

$$L^*_{\alpha(\mathbf{z})}\mathcal{X}_{\alpha(\mathbf{z})}(\mathbf{z}) \underset{\text{m.s.}}{=} \varepsilon(\mathbf{z}), \quad \forall \mathbf{z} \in \mathbb{R}^n, \tag{2.13}$$

where

$$X_{\alpha(\cdot)}(\varphi) \underset{\text{m.s.}}{=} \int_{\mathbb{R}^n} \mathcal{X}_{\alpha(\mathbf{z})}(\mathbf{z})\varphi(\mathbf{z})d\mathbf{z}, \quad \forall \varphi \in H^{\alpha(\cdot)}(\mathbb{R}^n). \tag{2.14}$$

Theorem 4 *Under the conditions assumed in Proposition 3, the ordinary solution to Equation (2.13) is Hölder continuous in the mean-square sense and satisfies, for every $\mathbf{z} \in \mathbb{R}^n$ and every $\mathbf{h} \in (0,1)^n$,*

$$\left(E\left[\mathcal{X}_{\alpha(\cdot)}(\mathbf{z}+\mathbf{h}) - \mathcal{X}_{\alpha(\cdot)}(\mathbf{z})\right]^2\right)^{1/2} \leq 2^{1/2}\|\mathbf{h}\|^{\underline{-\alpha}_{\epsilon_{\mathbf{h}}(\mathbf{z})}-n/2}, \tag{2.15}$$

where $\underline{-\alpha}_{\epsilon_{\mathbf{h}}(\mathbf{z})}$ represents the infimum of $-\alpha(\cdot)$ in a neighbourhood of $\mathbf{z}$ of radius $\|\mathbf{h}\|$.

Remark 1 *The proofs of Proposition 3 and Theorem 4 are provided in Ruiz-Medina, Anh and Angulo (2004a). From Theorem 4, the mean-square local Hölder exponent of the ordinary solution to Equation (2.13) changes in space according to the function $\underline{-\alpha}_{\epsilon_{\mathbf{h}}(\mathbf{z})} - n/2$.*

2.1 Pseudodifferential representation of variable order

In this subsection, the multifractional pseudodifferential representation of the class of random fields considered is obtained. Several examples of random fields satisfying this type of representation are then given. The operators $\widetilde{L}_{\alpha(\cdot)}$ and $L_{\alpha(\cdot)}$ are elliptic pseudodifferential operators belonging to the spaces $\boldsymbol{\mathcal{S}}^{\alpha(\cdot)}_{1,\delta}$ and $\boldsymbol{\mathcal{S}}^{-\alpha(\cdot)}_{1,\delta}$ respectively. The same assertion holds for $\widetilde{L}^*_{\alpha(\cdot)}$ and $L^*_{\alpha(\cdot)}$. Hence, they respectively belong to the spaces $\boldsymbol{\mathcal{S}}^{\alpha(\cdot)}_{\rho,\delta}$ and $\boldsymbol{\mathcal{S}}^{-\alpha(\cdot)}_{\rho,\delta}$ for $0 \leq \delta < \rho \leq 1$. Using Equation (2.2), the operators $L^*_{\alpha(\cdot)}$ and $\widetilde{L}_{\alpha(\cdot)}$ respectively admit the following representations:

$$\begin{aligned} L^*_{\alpha(\cdot)}f(\mathbf{x}) &= \int_{\mathbb{R}^n} e^{i\langle\mathbf{x},\boldsymbol{\xi}\rangle} l_{\alpha(\cdot)}(\mathbf{x},\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi})d\boldsymbol{\xi}, \; \forall f \in \left[H^{\alpha(\cdot)}(\mathbb{R}^n)\right]^*, \\ \widetilde{L}^*_{\alpha(\cdot)}\phi(\mathbf{x}) &= \int_{\mathbb{R}^n} e^{i\langle\mathbf{x},\boldsymbol{\xi}\rangle} \widetilde{l}_{\alpha(\cdot)}(\mathbf{x},\boldsymbol{\xi})\hat{\phi}(\boldsymbol{\xi})d\boldsymbol{\xi}, \; \forall \phi \in H^{\alpha(\cdot)}(\mathbb{R}^n). \end{aligned} \tag{2.16}$$

The operators $L^*_{\alpha(\cdot)}$ and $\widetilde{L}^*_{\alpha(\cdot)}$ respectively satisfy

$$\begin{aligned} L^*_{\alpha(\cdot)} &= l_{\alpha(\cdot)}\left(\cdot, \mathcal{L}\right), \\ \widetilde{L}^*_{\alpha(\cdot)} &= \widetilde{l}_{\alpha(\cdot)}\left(\cdot, \mathcal{L}\right), \end{aligned} \tag{2.17}$$

where $\mathcal{L}$ represents the linear self-adjoint differential operator on $L^2(\mathbb{R}^n)$ defined as

$$\mathcal{L} = \left(\mathcal{L}_{x_1}, \ldots, \mathcal{L}_{x_n}\right) = \left(-i\partial/\partial x_1, \ldots, -i\partial/\partial x_n\right).$$

Therefore, the following two pseudodifferential equations of variable order are defined:

$$l_{\alpha(\cdot)}\left(\cdot, \mathcal{L}\right)\mathcal{X}_{\alpha(\cdot)} \underset{\text{m.s}}{=} \varepsilon, \;\; \underline{-\alpha} > n/2,$$

$$\widetilde{l}_{\alpha(\cdot)}\left(\cdot, \mathcal{L}\right)\widetilde{\mathcal{X}}_{\alpha(\cdot)} \underset{\text{m.s}}{=} \varepsilon, \;\; \underline{\alpha} > n/2.$$

In the case where $\alpha(\cdot)$ is constant, we have that $L_{\alpha(\cdot)}$ and $\widetilde{L}_{\alpha(\cdot)}$ are fractal pseudodifferential operators in the sense introduced by Triebel (1997, Chapter V).

2.2 Examples

Example 1

Let $\mathcal{X}_{-\gamma(\cdot)}$ be defined as the mean-square solution to the following equation:

$$(-\Delta)^{\gamma(\cdot)/2}\mathcal{X}_{-\gamma(\cdot)} \underset{\text{m.s.}}{=} \varepsilon, \tag{2.18}$$

where $\gamma(\cdot) \in \mathcal{B}^{\infty}(\mathbb{R}^n)$, ε denotes Gaussian white noise, $(-\Delta)$ denotes the negative Laplacian, and $\alpha(\cdot) = -\gamma(\cdot)$. In this case,

$$L^{*}_{-\gamma(\cdot)} = (-\Delta)^{\gamma(\cdot)/2} = l_{-\gamma(\cdot)}(\cdot, \mathcal{L}) = \left(\mathcal{L}^2\right)^{\gamma(\cdot)/2} = ((i\mathcal{L})(-i\mathcal{L}))^{\gamma(\cdot)/2}.$$

Thus, $L^{*}_{-\gamma(\cdot)} = L_{-\gamma(\cdot)}$, and $l_{-\gamma(\cdot)}$ is real-valued. From Equation (2.16), $(-\Delta)^{\gamma(\cdot)/2}$ is defined as

$$(-\Delta)^{\gamma(\mathbf{x})/2}\phi(\mathbf{x}) = \int_{\mathbb{R}^n} e^{i\langle \mathbf{x},\boldsymbol{\xi}\rangle}|\boldsymbol{\xi}|^{\gamma(\mathbf{x})}\hat{\phi}(\boldsymbol{\xi})d\boldsymbol{\xi}. \tag{2.19}$$

In the case where $\gamma(\mathbf{x}) = n/2 + a(\mathbf{x})$, with $a(\cdot)$ belonging to $C^r(\mathbb{R}^n, (0,1))$ for some $r > \overline{a} = \sup_{\mathbf{x}\in\mathbb{R}^n} a(\mathbf{x})$, we have the multifractional Brownian motion model considered in Benassi *et al.* (1997).

Example 2

We now consider the pseudodifferential equation

$$(I-\Delta)^{\beta(\cdot)/2}\mathcal{X}_{-\beta(\cdot)} \underset{\text{m.s.}}{=} \varepsilon, \tag{2.20}$$

where $\beta(\cdot) \in \mathcal{B}^{\infty}(\mathbb{R}^n)$, $\alpha(\cdot) = -\beta(\cdot)$, and ε is white noise. Here,

$$L^{*}_{-\beta(\cdot)} = (I-\Delta)^{\beta(\cdot)/2} = l_{-\beta(\cdot)}(\cdot, \mathcal{L}) = \left(I - \mathcal{L}^2\right)^{\beta(\cdot)/2} = (I - (i\mathcal{L})(-i\mathcal{L}))^{\beta(\cdot)/2}.$$

From Equation (2.16), $(I-\Delta)^{\beta(\cdot)/2}$ is defined as

$$(I-\Delta)^{\beta(\mathbf{x})/2}\phi(\mathbf{x}) = \int_{\mathbb{R}^n} e^{i\langle \mathbf{x},\boldsymbol{\xi}\rangle}\left(1+|\boldsymbol{\xi}|^2\right)^{\beta(\mathbf{x})/2}\hat{\phi}(\boldsymbol{\xi})d\boldsymbol{\xi}. \tag{2.21}$$

Example 3

An extension of the fractional Riesz-Bessel motion introduced in Anh *et al.* (1999) to the variable order case is now formulated. This extension has as particular cases the above examples. We consider the equation

$$(I-\Delta)^{\beta(\cdot)/2}(-\Delta)^{\gamma(\cdot)/2}\mathcal{X}_{-(\beta(\cdot)+\gamma(\cdot))} \underset{\text{m.s.}}{=} \varepsilon, \tag{2.22}$$

where $\alpha = -(\beta(\cdot)+\gamma(\cdot))$. In this case,

$$\begin{aligned} l_{-(\beta(\cdot)+\gamma(\cdot))}(\mathbf{x},\boldsymbol{\xi}) &= \left(1+|\boldsymbol{\xi}|^2\right)^{\beta(\mathbf{x})/2}|\boldsymbol{\xi}|^{\gamma(\mathbf{x})}, \quad \forall \mathbf{x},\boldsymbol{\xi}\in\mathbb{R}^n, \\ \widetilde{l}_{-(\beta(\cdot)+\gamma(\cdot))}(\mathbf{x},\boldsymbol{\xi}) &= \left(1+|\boldsymbol{\xi}|^2\right)^{-\beta(\mathbf{x})/2}|\boldsymbol{\xi}|^{-\gamma(\mathbf{x})}, \quad \forall \mathbf{x},\boldsymbol{\xi}\in\mathbb{R}^n. \end{aligned} \tag{2.23}$$

Example 4

The next family of models includes the previous ones and provides an extension of the rational covariance family of fixed order (see Ramm 1990 and Angulo *et al.* 2000). Let $\mathcal{X}_{-((q(\cdot)-p(\cdot))s(\cdot))}$ be the mean-square solution to the following equation:

$$Q_{q(\cdot)}(\mathcal{A})\mathcal{X}_{-((q(\cdot)-p(\cdot))s(\cdot))} = P_{p(\cdot)}(\mathcal{A})\varepsilon, \tag{2.24}$$

where Q and P are elliptic polynomials of variable orders $q(\cdot)/2$ and $p(\cdot)/2$, respectively, and $\mathcal{A}$ can be a self-adjoint elliptic differential operator of fixed order ($s(\cdot)\equiv s$), or an elliptic pseudodifferential operator in the space $\boldsymbol{\mathcal{S}}^{s(\cdot)}_{\rho,\delta}$. Here, as before, $p(\cdot), q(\cdot), s(\cdot) \in \mathcal{B}^{\infty}(\mathbb{R}^n)$, and $\alpha(\cdot) = (-(q(\cdot)-p(\cdot))s(\cdot)/2)$. Also, the corresponding operators $\mathcal{A}^{\tau}, \tau\in\mathbb{R}$, are understood as fractional powers of self-adjoint elliptic differential operators (see Triebel 1978, Section 1.15, or Triebel 1997, pp. 213-215). For example, taking

$$Q(x) = a_n(ix)^{q_n} + a_{n-1}(ix)^{q_{n-1}} + \ldots + a_0(ix)^{q_0}, \; q_0 < \ldots < q_{n-1} < q_n$$

and $\mathcal{A} = -i\partial/\partial x \equiv -i\mathcal{D}$, then

$$Q(-i\mathcal{D}) = a_n\mathcal{D}^{q_n} + a_{n-1}\mathcal{D}^{q_{n-1}} + \ldots + a_0\mathcal{D}^{q_0}, \; q_0 < \ldots < q_{n-1} < q_n.$$

3 Random fields of variable regularity order on multifractal domains

Let μ be a Borel measure in $\mathbb{R}^n$. The singularity exponent $\alpha(\cdot)$ of μ can be estimated as follows: For each $\mathbf{z} \in \mathbb{R}^n$,

$$\alpha(\mathbf{z}) = \lim_{l \longrightarrow \infty} \frac{\log\left(\mu(Q_{2^{-l}}(\mathbf{z}))\right)}{-l\log 2}. \tag{3.1}$$

where $Q_{2^{-l}}(\mathbf{z}))$ is a cube in $\mathbb{R}^n$ centred at $\mathbf{z}$ with the sides parallel to the axes of coordinates, and with side length 2^{-l}. If the above limit does not exists, $\alpha(\cdot)$ is considered undefined.

Definition 5 *A Borel measure μ in $\mathbb{R}^n$ is said to be multifractal if the scaling behaviour of μ at $\mathbf{z}$ depends on the location $\mathbf{z}$, that is, if the singularity exponent $\alpha(\mathbf{z})$ varies as $\mathbf{z}$ varies.*

In this section we consider a characterization of fractional Sobolev spaces of variable order on multifractal domains via the atomic decomposition. This characterization allows the interpretation of these spaces in terms of the trace operators on multifractal domains.

We consider a mulifractal measure $\mu_{\mathcal{M}}$ with singularity exponent $\alpha(\cdot)$ defining the local dimension of a domain $\mathcal{M}$, which in this case is said to be a multifractal domain. The restriction to the multifractal domain $\mathcal{M}$ of functions in the space $H^s(\mathbb{R}^n)$, with $s > \frac{n-\overline{\alpha}}{2}$ and with $\overline{\alpha} < n$, can be defined in terms of the trace operator given in Proposition 6 below.

Lemma 5 *The multifractal measure μ defined as in Equation* (3.1) *satisfies that there exist two positive constants c_1 and c_2 such that*

$$c_1 2^{-l\alpha(2^{-\nu}m)} \leq \mu\left(Q_{2^{-l}}(2^{-\nu}m)\right) \leq c_2 2^{-l\alpha(2^{-\nu}m)}, \tag{3.2}$$

that is,

$$\mu\left(Q_{2^{-l}}(2^{-\nu}m)\right) \sim 2^{-l\alpha(2^{-\nu}m)}.$$

Proposition 6 *Let $\mathcal{M}$ be a compact multifractal domain with variable singularity exponent $\alpha(\cdot)$ defining its local dimension. Then, the following identity holds:*

$$\mathrm{tr}_{\mathcal{M}}\left(H^{\frac{n-\alpha(\cdot)}{2}}(\mathbb{R}^n)\right) = L^2(\mathcal{M}).$$

Corollary 7 *Under the conditions assumed in the previous proposition, the following identity holds:*

$$\mathrm{tr}_{\mathcal{M}}\left(H^{\widetilde{s(\cdot)+\frac{n-\alpha(\cdot)}{2}}}(\mathbb{R}^n)\right) = H^{\widetilde{s(\cdot)}}(\mathcal{M}). \tag{3.3}$$

The proofs of Lemma 5, Proposition 6 and Corollary 7 are provided in Ruiz-Medina, Anh and Angulo (2004b), where the function $\widetilde{\alpha}(\mathbf{x})$ and the corresponding space $H^{\widetilde{\alpha}(\cdot)}(\mathbb{R}^n)$ are defined in relation to the atomic decomposition of fractional Sobolev spaces. Corollary 7 provides the relationship between the multifractal geometry and regularity of functions in fractional Sobolev spaces of variable order. Indeed, the two functions $s(\cdot)$ and $\alpha(\cdot)$ respectively define the order of regularity of functions restricted to a multifractal domain, and the local dimension of such multifractal domain. The norms considered on the spaces $H^{\widetilde{s(\cdot)+\frac{n-\alpha(\cdot)}{2}}}(\mathbb{R}^n)$ and $H^{\widetilde{s}(\cdot)}(\mathcal{M})$ are based on the local variation of functions $s(\cdot)$ and $\alpha(\cdot)$ on the cubes defining the support of atoms involved in their atomic decompositions (Ruiz-Medina, Anh and Angulo 2004b).

Definition 6 *Let $\mathcal{M}$ be a compact multifractal domain with local dimension given by the singularity exponent $\alpha(\cdot)$, and let $X_{\widetilde{\beta}(\cdot)}$ be an FGRFVO with variable order $\widetilde{\beta}(\cdot)$. For $\widetilde{\beta}(\cdot) = \widetilde{s(\cdot)+\frac{n-\alpha(\cdot)}{2}}$, the restriction $X^{\mathcal{M}}_{\widetilde{s(\cdot)}}$ of $X_{\widetilde{\beta}(\cdot)}$ to a compact multifractal domain $\mathcal{M}$ is defined as*

$$X^{\mathcal{M}}_{\widetilde{s(\cdot)}}(f) = X_{\widetilde{\beta}(\cdot)}(f), \quad \forall f \in H^{-\widetilde{\beta}(\cdot),\mathcal{M}}(\mathbb{R}^n).$$

We refer to $X^{\mathcal{M}}_{\widetilde{s(\cdot)}}$ as the mean-square trace of $X_{\widetilde{\beta}(\cdot)}$ on the compact multifractal domain $\mathcal{M}$.

It then follows that

$$\begin{aligned} R_{X^{\mathcal{M}}_{\widetilde{s(\cdot)}}} &= J_{\mathcal{M}} J^*_{\mathcal{M}}, \\ R_{\widetilde{X}^{\mathcal{M}}_{\widetilde{s(\cdot)}}} &= J'_{\mathcal{M}} [J'_{\mathcal{M}}]^*. \end{aligned} \tag{3.4}$$

3.1 Hölder continuity

The class of corresponding FGRFVOs is included in the class of mean-square Hölder continuous random fields in the case where $\underline{\widetilde{s(\cdot)}} > \overline{\widetilde{\alpha(\cdot)/2}}$. The associated singularity spectrum is then defined in terms of the function $\widetilde{s(\cdot)} - \widetilde{\alpha(\cdot)/2}$. In the Gaussian case, such function also provides information on the Hölder spectrum of the sample paths.

Under the pseudoduality condition on multifractal domains and for $\underline{\widetilde{s(\cdot)}} > \overline{\widetilde{\alpha(\cdot)/2}}$, there exists a unique mean-square Hölder continuous ordinary random field $\mathcal{X}^{\mathcal{M}}_{\widetilde{s(\cdot)}}$ satisfying

$$L^*_{\mathcal{M}} \mathcal{X}^{\mathcal{M}}_{\widetilde{s(\cdot)}}(\mathbf{z}) \underset{\text{m.s.}}{=} \varepsilon_{\mathcal{M}}(\mathbf{z}), \quad \forall \mathbf{z} \in \mathcal{M}. \tag{3.5}$$

Its local quadratic variation is defined by the function $\widetilde{s(\cdot)} - \widetilde{\alpha(\cdot)/2}$ (Ruiz-Medina, Anh and Angulo 2004b). The mean-square Hölder spectrum (or singularity spectrum in the second-order moment sense) of the random field $\mathcal{X}^{\mathcal{M}}_{\widetilde{s(\cdot)}}$ is then defined as the function $d(\cdot)$ which relates the local Hölder exponent γ to the Hausdorff dimension of the set of points where the function $\widetilde{s(\cdot)} - \widetilde{\alpha(\cdot)/2}$ takes such value γ.

Under the pseudoduality condition on multifractal domains and in the case where function $s(\cdot)$ is positive, the covariance operator $R_{X^{\mathcal{M}}_{\widetilde{s(\cdot)}}}$ of $X^{\mathcal{M}}_{\widetilde{s(\cdot)}}$ is compact, and the point spectrum $\left\{\lambda_k\left(R_{\widetilde{X}^{\mathcal{M}}_{\widetilde{s(\cdot)}}}\right)\right\}_{k\in\mathbb{N}}$ of $R_{\widetilde{X}^{\mathcal{M}}_{\widetilde{s(\cdot)}}}$ satisfies

$$ck^{\frac{2\widetilde{s(\cdot)}}{\alpha(\cdot)}} \leq \lambda_k\left(R_{\widetilde{X}^{\mathcal{M}}_{\widetilde{s(\cdot)}}}\right) \leq Ck^{\frac{2\widetilde{s(\cdot)}}{\alpha(\cdot)}}, \; k \in \mathbb{N}$$

(see Ruiz-Medina, Anh and Angulo 2004b).

3.2 Special cases

In the case where $R_{\widetilde{X}^{\mathcal{M}}_{\widetilde{s(\cdot)}}}$ is local, the operator $J'_{\mathcal{M}}$ admits a differential representation on the multifractal domain $\mathcal{M}$. Such a representation is obtained from the compactness of embeddings between fractional Sobolev spaces of variable order on $\mathcal{M}$. The operator $J'_{\mathcal{M}}$ then admits a representation of the form

$$\sum_{|\widetilde{\sigma}(\cdot)| \leq [\widetilde{s}(\cdot)]^-} a^{\mathcal{M}}_{\widetilde{\sigma}(\cdot)}(\cdot) \langle D.\rangle^{\widetilde{s}_+(\cdot)/2} \langle D.\rangle^{\widetilde{\sigma}(\cdot)} \cdot \mathrm{tr}_{\mathcal{M}} \underset{\text{m.s.}}{=} \varepsilon_{\mathcal{M}}, \tag{3.6}$$

where $a^{\mathcal{M}}_{\widetilde{\sigma}(\cdot)}(\cdot)$ denotes a distribution with compact support $\mathcal{M}$, $\widetilde{s}(\cdot) = [\widetilde{s}(\cdot)]^- + \widetilde{s}_+(\cdot)$, $[s]^-$ denotes the the integer part of s and $\widetilde{\sigma}(\cdot)$ is an interger-valued function.

For example, $(-\Delta)^{\widetilde{\gamma}(\cdot)/2} \cdot \mathrm{tr}_{\mathcal{M}}$ belongs to the class defined in terms of the operator $J'_{\mathcal{M}}$, where $(-\Delta)^{\widetilde{\gamma}(\cdot)/2}$ is the negative Laplacian of variable order $\widetilde{\gamma}(\cdot)/2$, with $\gamma \in \mathcal{B}^\infty(\mathbb{R}^n)$, and $\mathrm{tr}_{\mathcal{M}}$ denotes the trace operator on $\mathcal{M}$ in the sense given by Corollary 7. Hence,

$$(-\Delta)^{\widetilde{\gamma}(\cdot)/2} \mathrm{tr}_{\mathcal{M}} \mathcal{X}_{\widetilde{\gamma}(\cdot)} \underset{\text{m.s.}}{=} \varepsilon_{\mathcal{M}} \tag{3.7}$$

can be considered as a particular case of the models given by (3.5). Also the following equation

$$\langle D.\rangle^{\widetilde{s}(\cdot)} (-\Delta)^{\widetilde{\gamma}(\cdot)/2} \mathrm{tr}_{\mathcal{M}} \mathcal{X}_{\widetilde{\gamma}(\cdot)+\widetilde{s}(\cdot)} \underset{\text{m.s.}}{=} \varepsilon_{\mathcal{M}}$$

provides another special case of (3.5). In general, in terms of the restriction to $\mathcal{M}$ of an elliptic rational function Q/P, with variable order $\widetilde{q}(\cdot)/2 - \widetilde{p}(\cdot)/2$, of an elliptic self-adjoint differential operator $\mathcal{A}$ of variable order $\widetilde{s}(\cdot)$, special cases of (3.5) can be defined as

$$Q_{\widetilde{q}(\cdot)}(\mathcal{A}) \mathrm{tr}_{\mathcal{M}} \mathcal{X}_{-((\widetilde{q}(\cdot)-\widetilde{p}(\cdot))\widetilde{s}(\cdot))} = P_{\widetilde{p}(\cdot)}(\mathcal{A}) \varepsilon_{\mathcal{M}}. \tag{3.8}$$

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